Versal deformations in generalized flag manifolds

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Abstract

We find explicit miniversal deformations of flags in the generalized flag manifolds, with regard a natural equivalence relation defined by the group action that keeps fixed the reference flag.

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Introduction

Given a smooth manifold $\mathcal{M}$ and an element of $x \in \mathcal{M}$, a versal deformation of x with regard an equivalent relation defined by a group action is a concept introduced by Arnold in [1] and subsequently developed in the study of local perturbations. Although Arnold's theory has been stated in a very general formulation, in practice, almost all explicit versal deformations have been calculated in open sets of $\mathbb{R}^n$. In [4], versal deformations in orbit spaces are studied and a description of them in terms of a system of equations is obtained ([4]-Theorem 2.6). Nevertheless, these equations are, in general, unable to be solved explicitly. In other terms, it is needed for each orbit space, a particular treatment of these equations.

In this paper we apply the results of [4] to generalized flag manifolds, which are a particular kind of orbit space that generalize the Grassmann manifold. More precisely, we fix a reference flag (a chain of subspaces $F_1 \subset \cdots \subset F_k$ of a $\mathbb{F}$-vector space F) and we consider the set of subflags of it whose elements have fixed dimension. This set is a smooth manifold with an orbit space structure similar to that of the Grassmann manifold (see, for example, [2]). The generalized flag manifold arises in a natural way while studying the topology of the set of A-invariant and (C, A)-invariant subspaces. More precisely in [2] and [3] it is proved the existence of deformation retracts of the set of A-invariant subspaces and (C, A)-invariant subspaces, respectively, on a generalized flag manifold (when the discrete invariants of the restrictions are fixed). The natural equivalence relations defined in the set of A and in the set of (C, A)-invariant subspaces induce an equivalence relation in the generalized flag manifold as follows: two flags are equivalent if there exists an automorphism keeping fixed the reference flag such that sends the elements of the first flag to the elements of the second one.

The organization of this paper is the following one. In section 1 we recall the basic definitions and results on the generalized flag manifold and versal deformations in orbit spaces. In section 2 we study the classification of subflags of a given flag. Finally, in section 3 we find an explicit miniversal deformation of a subflag with regard to the above classification.

In this paper we make use of the following notation. $\mathbb{F}$ is the field of either the complex or the real numbers. If E is an $\mathbb{F}$-vector space, $\mathrm{Gr}_d(E)$ denotes the Grassmann manifold of d-dimensional subspaces of E. We say that a basis is *adapted* to a set of subspaces if one can obtain bases of those subspaces taking subsets of this basis. In particular, a basis adapted to a chain of subspaces $V_1 \subset \cdots \subset V_k$ is a basis of V_k obtained by extending successively bases of $V_1, V_2, \ldots$. We say that an n-tuple of integers $(k_1, \ldots, k_n)$ is a *partition* if $0 \leq k_i \leq k_{i+1}$ for all $i = 1, \ldots, n-1$. I_a means the $a \times a$-identity matrix and $0_{a,b}$ the $a \times b$-zero matrix.

1 Preliminaries

Let F be a fixed $\mathbb{F}$-vector space of dimension n. A flag of F is a chain of subspaces $V_1 \subset \cdots \subset V_k$. We are given a reference flag, $F_1 \subset \cdots \subset F_k$, that is fixed all along this note. A flag $V_1 \subset \cdots \subset V_k$ such that $V_i \subset F_i$, $i = 1, \ldots, k$, is said to be *a subflag* of $F_1 \subset \cdots \subset F_k$.

Let $s = (s_1, \ldots, s_k)$ be a partition with $s_1 \leq \cdots \leq s_k \leq n$. We say that $V_1 \subset \cdots \subset V_k$ is an *s-flag* if $\dim V_i = s_i$, $i = 1, \ldots, k$. Assume that $F_1 \subset \cdots \subset F_k$ is an r-flag with $r = (r_1, \ldots, r_k)$. If $s_i \leq r_i$ for $i = 1, \ldots, k$, the *generalized flag manifold*, $\mathrm{Flag}(r, s)$, is the set of s-subflags of the above reference r-flag. $\mathrm{Flag}(r, s)$ is diffeomorphic to the orbit space $\mathcal{M}(r, s)/\mathcal{M}(s, s)$, where $\mathcal{M}(r, s)$ is the set of full rank matrices of the block form $X = [X_{i,j}]$, being $X_{i,j}$ an $(r_i - r_{i-1}) \times (s_j - s_{j-1})$-matrix ($r_0 := 0$) with $X_{i,j} = 0$ if $j < i$, and defining in $\mathcal{M}(r, s)$ the action of $\mathcal{M}(s, s)$ by $(X, P) \mapsto XP$. We refer the reader to [2] for the corresponding developments.

We define in $\mathrm{Flag}(r, s)$ the following equivalence relation.

Definition 1.1 *$(V_1, \dots, V_k) \sim (V'_1, \dots, V'_k)$ if there exist $P \in \mathrm{Gl}(n)$ such that*

$$V'_i = P(V_i) \text{ and } P(F_i) = F_i \text{ for all } i = 1, \dots, k.$$

Let $\mathcal{G} = \{P \in \mathrm{Gl}(n) | P(F_i) = F_i\}$. It is clear that $\mathcal{G}$ is a subgroup of $\mathrm{Gl}(n)$ and that the classes of the above equivalence relation are the orbits of the subflags of $(F_1, \dots, F_k)$ with regard to the action of $\mathcal{G}$ on $\mathrm{Flag}(r, s)$ defined by

$$(P, (V_1, \dots, V_k)) \mapsto (P(V_1), \dots, P(V_k)).$$

Taking a basis of F adapted to $(F_1, \dots, F_k)$, one can check that $\mathcal{G}$ is $\mathcal{M}(r, r)$. For notational convenience, let us denote $\mathcal{M}(s, s)$ by Γ and $\mathcal{M}(r, s)$ by $\mathcal{M}$. It is clear that there exists a bijection between the orbits $\mathcal{G}(X\Gamma)$, $X\Gamma \in \mathcal{M}/\Gamma$, and the orbits $\mathcal{G}X\Gamma$ corresponding to the action on $\mathcal{M}$ defined by $((P, Q), X) \mapsto PXQ$, where in $\mathcal{G} \times \Gamma$ we define the group product $(P, Q)(P', Q') := (P'P, QQ')$.

In the next sections we exhibit a miniversal deformation of any given element $X\Gamma$ of the orbit space $\mathcal{M}/\Gamma$. Our notation here is especially adapted to the notation and context of the previous work [4]. The reader may find there a definition and results relevant for our present purposes. The key result that we apply in this note is the following theorem of [4].

Theorem 1.2 *A miniversal deformation of an orbit $X\Gamma$ in $\mathcal{M}/\Gamma$ is $(X+W)\Gamma$ where $W \in \mathcal{W}$, being $\mathcal{W}$ a neighbourhood of the origin of the set of matrices $W \in \overline{\mathcal{M}}$ such that*

$$\mathrm{trace}\,(PXW^*) = \mathrm{trace}\,(XQW^*) = 0$$

for all $P \in \overline{\mathcal{G}}$ and $Q \in \overline{\Gamma}$.

In section 3 we solve explicitly the equations $\mathrm{trace}\,(PXW^*) = \mathrm{trace}\,(XQW^*) = 0$ in the context of the generalized flag manifold. For this purpose it is needed a canonical representant of the orbit $\mathcal{G}X\Gamma$ in terms of a complete set of invariants of $X\Gamma$. This is the goal of section 2.

2 Classification of the subflags of a reference flag

Now we will find a canonical element of the orbit $\mathcal{G}X\Gamma$. One can easily prove that by means of the action of $\mathcal{G}$ and Γ, one can reduce the matrix X to a permutation matrix of $\mathcal{M}$. However, we are specially interested in describing the positions of the 1's of this permutation matrix in terms of geometric invariants of the flag represented by X. In order to do that, we interpret the columns of X as a basis of F adapted to the chain of subspaces of the corresponding flag. This leads us to consider the following diagram

$$
\begin{array}{ccccccccc}
V_1 = V_1 \cap F_1 & \subset & V_2 \cap F_1 & \subset & \cdots & \subset & V_k \cap F_1 & \subset & F_1 \\
\cap & & \cap & & & & \cap & & \cap \\
V_2 & = & V_2 \cap F_2 & \subset & \cdots & \subset & V_k \cap F_2 & \subset & F_2 \\
& & \cap & & & & \cap & & \cap \\
& & V_3 & = & \cdots & \subset & V_k \cap F_3 & \subset & F_3 \\
& & & \ddots & & & \vdots & & \vdots \\
& & & & & & V_k & \subset & F_k
\end{array} \tag{1}
$$

Defining $F_{k+1} = V_{k+1} = F$ and $F_0 = V_0 = \{0\}$, we can write for $k \geq i \geq j \geq 0$

$$
\begin{array}{ccc}
V_i \cap F_j & \subset & V_{i+1} \cap F_j \\
\cap & & \cap \\
V_i \cap F_{j+1} & \subset & V_{i+1} \cap F_{j+1}
\end{array}
$$

where, obviously, $V_i \cap F_j = (V_{i+1} \cap F_j) \cap (V_i \cap F_{j+1})$. Let $E_{i,j}$ be a subspace such that

$$V_i \cap F_j = E_{i,j} \oplus (V_{i-1} \cap F_j + V_i \cap F_{j-1})$$

and let $e_{i,j}$ be a basis of $E_{i,j}$. Arranging the vectors of $\bigcup_{i,j} e_{i,j}$ in order to obtain bases of V_k and F_k we have that

$$
\begin{array}{ll}
\{e_{1,1};\ e_{2,1}, e_{2,2}; \ldots; e_{k,1}, \ldots, e_{k,k}\} & \text{is a basis of } V_k \text{ adapted to } (V_1, \ldots, V_k) \\
\{e_{1,1}, \ldots, e_{k,1};\ e_{2,2}, \ldots, e_{k,2}; \ldots; e_{k+1,k+1}\} & \text{is a basis of } F_k \text{ adapted to } (F_1, \ldots, F_k)
\end{array} \tag{2}
$$

Arranging in columns the coefficients of the elements of the first basis with regard the second one can check that we obtain the following matrix

(3) $X = [X_{j,i}]$, $0 < j \leq i \leq k+1$, with $X_{j,i} = 0$ if $i < j$, and for $i \geq j$,

$$
X_{j,i} = \begin{bmatrix}
0_{\beta_{i,j},\alpha_{i,j}} & 0 & 0 \\
0 & I_{s_{i,j}} & 0 \\
0 & 0 & 0_{\overline{\beta}_{i,j},\overline{\alpha}_{i,j}}
\end{bmatrix}
$$

where

$$s_{i,j} := \dim E_{i,j} = \dim V_i \cap F_j - \dim V_{i-1} \cap F_j - \dim V_i \cap F_{j-1} + \dim V_{i-1} \cap F_{j-1}$$

and

$$\alpha_{i,j} := s_{i-1,j} + s_{i-2,j} + \cdots s_{j,j} \text{ or } 0 \text{ if } i = j$$

$$\overline{\alpha}_{i,j} := s_{k,j} + \cdots + s_{i+1,j}$$

$$\beta_{i,j} := s_{i,j-1} + \cdots + s_{i,1} \text{ or } 0 \text{ if } j = 0$$

$$\overline{\beta}_{i,j} := s_{i,i} + \cdots + s_{i,j+1}$$

This proves the following theorem

Theorem 2.1 *Let $(V_1, \ldots, V_k)$ be a subflag of $(F_1, \ldots, F_k)$ represented by $X \in \mathcal{M}$. With the above notations we have that*

(i) The matrix defined in (3) is a canonical form of the orbit $\mathcal{G}X\Gamma$.
(ii) $s_{i,j}$ for $0 < j \le i \le k$, is a complete set of invariants of $\mathcal{G}X\Gamma$.
(iii) $\dim V_i \cap F_j$ for $0 < j \le i \le k$, is a complete set of invariants of $\mathcal{G}X\Gamma$.

Moreover, there exists a bijection between $\mathrm{Flag}(r,s)/\mathcal{G}$ and the set of integers $s_{i,j}$ satisfying for $0 \le i \le k$ the conditions

$$s_i - s_{i-1} = s_{i,1} + \cdots + s_{i,i}$$

$$r_i - r_{i-1} = s_{i,i} + \cdots + s_{k,i}$$

Besides last theorem, the construction of the canonical bases (2), has the following consequence.

Theorem 2.2 *Let $(V_1(t), \ldots, V_k(t))$ be a smooth family of flags of the same class parameterized by a contractible manifold M. Then, there exists a smooth family of bases of $V_i(t)$, $t \in M$, defined as in (2). In particular, if $X(t)$ is a smooth family of matrices of $\mathcal{M}$ of the same class parameterized by M, there exist smooth families $P(t)$ and $Q(t)$ of $\mathcal{G}$ and Γ, respectively such that $P(t)X(t)Q(t)$ is the matrix (3).*

3 Versal deformations in the generalized flag manifold

In this section we apply theorem 1.2 in the context of the generalized flag manifold. For this purpose, we solve the equations $\mathrm{trace}\,(PXW^*) = \mathrm{trace}\,(XQW^*) = 0$ where $X \in \mathcal{M}, P \in \mathcal{G}$ and $Q \in \Gamma$. Since the map $X \mapsto PXQ$ is a diffeomorphism, we can assume that X is a particular element of the orbit $\mathcal{G}X\Gamma$. So, we assume that X is the canonical matrix (3) described in the last section.

We have that $XQ = [\overline{Q_{i,j}}]$ where

$$\overline{Q_{i,j}} = \begin{bmatrix} Q_{i,j}^1 \\ Q_{i,j}^2 \\ \dots \end{bmatrix},$$

with $Q_{i,j}^h$ being the corresponding rows of $Q_{i,j}$ if $h \leq j$, and $Q_{i,j}^h = 0$ if $h > j$.

On the other side, $PX = [\overline{P_{i,j}}]$ where $\overline{P_{i,j}} = \begin{bmatrix} P_{i,j}^1 \ P_{i,j}^2 \ \dots \end{bmatrix}$, with $P_{i,j}^h$ being the corresponding columns of $P_{i,j}$ if $h \geq j$, and $P_{i,j}^h = 0$ if $h < j$.

Notice that the set of matrices Y with a fixed zero structure and having in the rest of the entries different, independent parameters is a vector space of dimension the number of parameters, and its orthogonal is the set of matrices of the same size, having different, independent parameters in the zero entries of Y and 0 in the nonzero entries of Y.

Therefore, taking into account the form of the matrices PX and XQ, we have that $\operatorname{trace}(PXW^*) = \operatorname{trace}(XQW^*) = 0$ implies that

(4) $W = [W_{j,i}], 0 < j \leq i \leq k+1$, with $W_{j,i} = 0$ if $i < j$, and for $i \geq j$, decomposing into blocks $W_{j,i}$ as in (3),

$$W_{j,i} = \begin{bmatrix} 0_{\beta_{i,j},\alpha_{i,j}} & 0 & 0 \\ 0 & 0_{s_{i,j}} & 0 \\ Y_{\overline{\beta}_{i,j},\alpha_{i,j}} & 0 & 0_{\overline{\beta}_{i,j},\overline{\alpha}_{i,j}} \end{bmatrix}$$

with $Y_{\overline{\beta}_{i,j},\alpha_{i,j}}$ a full parameter $\overline{\beta}_{i,j} \times \alpha_{i,j}$-matrix.

This proves the following theorem

Theorem 3.1 *With the above notation, the set of flags $(X + W)\Gamma$ where X and W are as in (3) and (4), respectively, is a miniversal deformation of the flag $X\Gamma$.*

Last theorem allows us to compute the dimension of the orbit of a given flag. In particular, stable flags (flags having the codimension of its orbit equal to 0) are characterized through the following corollary.

Corollary 3.2 *The flag $(V_1, \dots, V_k)$ is stable if and only if $\overline{\beta}_{i,j} = 0$ for all $0 < j \leq i \leq k$. In particular, all the stable flags are equivalent.*

Moreover, the versal deformation obtained in the last theorem, allows us to compute bifurcation diagrams as it is shown in the following example

Example 3.3 Let $r = (2, 3, 4)$ and $s = (1, 2, 3)$. We have four possible flag classes represented by the canonical matrices

$$(A)\quad \begin{bmatrix} 1\ 0\ 0 \\ 0\ 0\ 0 \\ \quad 1\ 0 \\ \qquad 1 \end{bmatrix} \qquad (B)\quad \begin{bmatrix} 1\ 0\ 0 \\ 0\ 1\ 0 \\ \quad 0\ 0 \\ \qquad 1 \end{bmatrix} \qquad (C)\quad \begin{bmatrix} 1\ 0\ 0 \\ 0\ 0\ 1 \\ \quad 1\ 0 \\ \qquad 0 \end{bmatrix} \qquad (D)\quad \begin{bmatrix} 1\ 0\ 0 \\ 0\ 1\ 0 \\ \quad 0\ 1 \\ \qquad 0 \end{bmatrix}$$

The stable flags are the flags of the class of (A) and, on the other hand, a miniversal deformation of the element (D) is, according to the last theorem,

$$X + T_X(\mathcal{G}X\Gamma)^{\perp} = \{ \begin{bmatrix} 1\ 0\ 0 \\ 0\ 1\ 0 \\ \quad x\ 1 \\ \qquad y \end{bmatrix} \}$$

Therefore, the bifurcation diagram of (D) is given by the following partition of the space of parameters, according to his adjacent classes.

$$\begin{aligned} &\text{for } x = 0 \text{ and } y \neq 0, \text{ class of } (B) \\ &\text{for } x \neq 0 \text{ and } y = 0, \text{ class of } (C) \\ &\text{for } x \neq 0 \text{ and } y \neq 0, \text{ class of } (A) \end{aligned}$$

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